The dimension of the Incipient Infinite Cluster

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Overview

- Percolation on $\mathbb{Z}^d$ (for $d > 10$, 'high-dimensional')
- Definition Incipient Infinite Cluster (IIC)
- Theorems + idea of proof. Mass dimension of IIC is 4 and volume growth exponent of IIC is 2, a.s..

Percolation on $\mathbb{Z}^d$

Consider nearest-neighbour Bernoulli bond percolation on $\mathbb{Z}^d$, with parameter $p \in [0, 1]$ and measure $\mathbb{P}_p$.

Definition. Connected vertices.

$$\{x \leftrightarrow y\} := \{x \text{ connected to } y \text{ by a path of } \textit{open} \text{ edges } \}$$

Definition. Open cluster of $x \in \mathbb{Z}^d$.

$$\mathcal{C}(x) := \left\{ y \in \mathbb{Z}^d : x \leftrightarrow y \right\}$$

Definition. Critical probability.

$$p_c := \inf \{ p : \mathbb{P}_p (|\mathcal{C}(0)| = \infty) > 0 \}$$

What happens at the critical probability?

Phase transition in p_c:

$$\mathbb{P}_p\left(\exists x \in \mathbb{Z}^d \text{ s.t. } |\mathcal{C}(x)| = \infty\right) = \begin{cases} 0 & \text{if } p < p_c \\ 1 & \text{if } p > p_c. \end{cases}$$

What happens if $p = p_c$?

$$\mathbb{P}_{p_c}(\ldots) = 0 \text{ if } d = 2 \text{ or } d > 10.$$

Informally, the Incipient Infinite Cluster (IIC) is the critical cluster $\mathcal{C}(0)$ "conditioned on the event that $|\mathcal{C}(0)| = \infty$".

Intermezzo: high dimensions

Definition. High dimensions.

The model is called *high-dimensional* if the dimension d of $\mathbb{Z}^d$ is > 6 and satisfies the following. There are constants $c, C > 0$ s.t. for all $x, y \in \mathbb{Z}^d$

$$c \cdot \|x - y\|^{2-d} \leq \mathbb{P}_{p_c}(x \leftrightarrow y) \leq C \cdot \|x - y\|^{2-d}.$$

True for $d > 18$ [e.g. Hara, 2008] and $d > 10$ [vd Hofstad, Fitzner, 2015]. Believed to be true for $d > 6$.

Incipient Infinite Cluster

Recall, $\mathbb{P}_{p_c}(|\mathcal{C}(0)| = \infty) = 0$ in h.d.. But now "condition on $|\mathcal{C}(0)| = \infty$". Need new probability measure to make this precise.

Definition IIC measure

For cylinder events E,

$$\mathbb{P}_{\mathrm{IIC}}(E) := \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E \mid 0 \leftrightarrow x).$$

In h.d., this can be extended to a well defined measure [Heydenreich, vd Hofstad, Hulshof, 2014].

Definition IIC.

$$\mathrm{IIC} := \mathcal{C}(0).$$

Cube of radius r intersected with IIC

$$Q_r \cap \text{IIC} := \left\{ x \in \mathbb{Z}^d : 0 \leftrightarrow x \text{ and } \|x\|_\infty \leq r \right\}$$

Random ball of radius r.

$$B_r := \left\{ x \in \mathbb{Z}^d : 0 \leftrightarrow x \text{ and } d_{\text{IIC}}(0, x) \leq r \right\}$$

Mass dimension of IIC

$$d_m(\text{IIC}) := \lim_{r \to \infty} \left(\frac{\log |Q_r \cap \text{IIC}|}{\log(r)} \right)$$

Volume growth exponent of IIC

$$d_f(\text{IIC}) := \lim_{r \to \infty} \left(\frac{\log |B_r|}{\log(r)} \right)$$

How (infinitely) large is IIC under $\mathbb{P}_{\text{IIC}}$?

Theorem 1 [C, 2015]

In h.d. ($d > 10$),

$$\mathbb{P}_{\text{IIC}}\left(d_m(\text{IIC}) := \lim_{r\to\infty}\left(\frac{\log|Q_r \cap \text{IIC}|}{\log(r)}\right) = 4\right) = 1.$$

So IIC is 4-dimensional with respect to the 'extrinsic' distance of the surrounding lattice $\mathbb{Z}^d$.

Theorem 2 [C, 2015]

In h.d. ($d > 10$),

$$\mathbb{P}_{\text{IIC}}\left(d_f(\text{IIC}) := \lim_{r\to\infty}\left(\frac{\log|B_r|}{\log(r)}\right) = 2\right) = 1.$$

So IIC is 2-dimensional with respect to the 'intrinsic' graph distance.

Proof idea Theorem 1. Upper bound.

Upper bound is not the problem. Follows from

$$\mathbb{E}_{\mathrm{IIC}}|Q_r \cap \mathrm{IIC}| \leq C \cdot r^4$$

[e.g. vd Hofstad, Járai, 2004], combined with Markov's inequality and Borel-Cantelli.

Proof idea Theorem 1. Lower bound.

For the lower bound, results from literature [Kozma, Nachmias, 2009, 2011; vd Hofstad, Sapozhnikov, 2014] are used to obtain (roughly):

$$d_{\mathrm{IIC}}(0, \partial Q_r) \approx r^2 \text{ and } |B_r| \approx r^2.$$

Furthermore, $Q_r \cap \mathrm{IIC} \supseteq B_{d_{\mathrm{IIC}}(0,\partial Q_r)}$. So

Heuristic

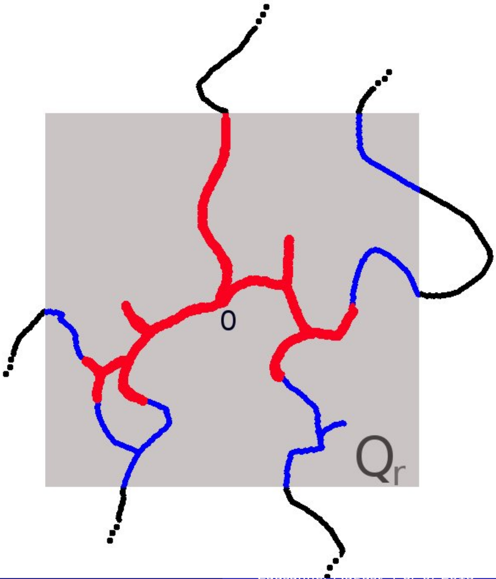
$$|Q_r \cap \mathrm{IIC}| \geq |B_{d_{\mathrm{IIC}}(0,\partial Q_r)}| \approx |B_{r^2}| \approx (r^2)^2 = r^4.$$

Remark: perhaps the most important ingredient from literature is

$$\mathbb{P}_{p_c}(0 \leftrightarrow \partial Q_r) \leq C \cdot \frac{1}{r^2}.$$

Visualisation $Q_r \cap \mathrm{IIC} \supseteq B_{d_{\mathrm{IIC}}(0,\partial Q_r)}$

IIC = red+blue+black ;
$Q_r \cap \mathrm{IIC}$ = red+blue;
$B_{d_{\mathrm{IIC}}(0,\partial Q_r)}$ = red.



Conjecture

Write $\partial Q_r = Q_r \backslash Q_{r-1}$ and let $0 \overset{Q_r}{\longleftrightarrow} x$ denote the event that 0 is connected to x by an open path that stays in Q_r.
Expect that $\mathbb{P}_{\mathrm{IIC}}-$ a.s.,

(i) $\# \{x \in \partial Q_r \mid 0 \longleftrightarrow x\} \asymp r^3$

(ii) $\# \left\{x \in \partial Q_r \mid 0 \overset{Q_r}{\longleftrightarrow} x\right\} \asymp r^2$

Lower bound for (ii) OK, upper bound for (i) OK.
Difficulty upper bound (ii): while we know that $\mathbb{P}_{p_c}(0 \longleftrightarrow x) \asymp \|x\|^{2-d}$, the behaviour of $\mathbb{P}_{p_c}\left(0 \overset{Q_r}{\longleftrightarrow} x\right)$ is not known accurately enough. In particular: depends on more than just the norm of x.

Summary

- Intuitively, the IIC is a critical cluster 'as it is becoming infinitely large'.
- In high dimensions ($d > 10$, expected $d > 6$), its mass dimension equals 4 and its volume growth exponent equals 2, $\mathbb{P}_{\text{IIC}}$-a.s..
- At the boundary of a cube of radius r, it seems there are $\approx r^3$ vertices that are connected to the origin but only $\approx r^2$ vertices that are connected to the origin by a path that stays inside the cube, $\mathbb{P}_{\text{IIC}}$-a.s..

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Extended proof scheme lower bound (including proofs in literature)

$$\mathbb{P}_{\mathrm{IIC}}\left(|Q_r \cap \mathrm{IIC}| \geq \lambda \cdot r^4\right) \leq C \cdot \frac{1}{\lambda}$$

$$\mathbb{P}_{\mathrm{IIC}}\left(0 \overset{\leq \epsilon r^2}{\longleftrightarrow} \partial Q_r\right) \leq C \cdot \sqrt{\epsilon}$$

$$\mathbb{P}_{\mathrm{IIC}}\left(|B_r| \leq \frac{1}{\lambda} r^2\right) \leq C \cdot \frac{1}{\lambda}$$

$$\mathbb{P}_{p_c}\left(0 \leftrightarrow \partial Q_r\right) \leq C \cdot \frac{1}{r^2}$$

$$\mathbb{P}_{p_c}\left(B_r \backslash B_{r-1} \neq \emptyset\right) \leq C \cdot \frac{1}{r}$$

$$\mathbb{E}_{p_c}\left(|B_r|\right) \leq C \cdot r$$

$$\mathbb{P}_{p_c}\left(|\mathcal{C}(0)| \geq r\right) \leq C \cdot \frac{1}{\sqrt{r}}$$